DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES

LESSON 14 UNIT 07

OBJECTIVES

STUDENTS WILL BE ABLE TO:

Solve the trigonometric problems using half-angle and double-angle identities.

KEY VOCABULARY:

- Half-angle identities
- Double-angle identities
- Sine, Cosine and Tangent

DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES

In dealing with trigonometry using angle sum and difference identities (for two angles), we can consider both the angles same and derive the **half-angle** and **double-angle** identities.

Half-angle means the given angle is halved and is related in some way.

Double-angle means the given angle is doubled and is related in some way.

DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES

Let's first derive the double-angle and half-angle identities.

Deriving the **double-angle identities**:

For deriving these identities, we need to know the following three angle sum identities:

- $\sin(\theta + \varphi) = \sin(\theta)\cos(\varphi) + \cos(\theta)\sin(\varphi)$ ... (i)
- $cos(\theta + \varphi) = \cos(\theta)\cos(\varphi) - \sin(\theta)\sin(\varphi)$ ... (ii)
- $\tan(\theta + \varphi) = \frac{\tan(\theta)+\tan(\varphi)}{1-\tan(\theta)\tan(\varphi)}$ ... (iii)

DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES

Consider (i) and put $\varphi = \theta$;

$$\rightarrow \sin(\theta + \theta) = \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta)$$

or, $$\boldsymbol{sin(2\theta) = 2\sin(\theta)\cos(\theta)} \quad \textbf{... (A)}$$

Consider (ii) and put $\varphi = \theta$;

$$\rightarrow \cos(\theta + \theta) = \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta)$$

or, $$\boldsymbol{cos(2\theta) = cos^2(\theta) - sin^2(\theta)} \quad \textbf{... (B)}$$

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Consider (iii) and put $\varphi = \theta$;

$$\rightarrow \tan(\theta + \theta) = \frac{\tan(\theta)+\tan(\theta)}{1-\tan(\theta)\tan(\theta)}$$

or, $$\boldsymbol{tan(2\theta) = \frac{2tan(\theta)}{1-tan^2(\theta)}} \quad \textbf{... (C)}$$

PROBLEM 1:

Show that: $\boldsymbol{cos(2\theta) = 2cos^2(\theta) - 1 = 1 - 2sin^2(\theta)}$

Since, $cos(2\theta) = cos^2(\theta) - sin^2(\theta)$

$\rightarrow cos(2\theta) = cos^2(\theta) - (1 - cos^2(\theta)) = cos^2(\theta) - 1 + cos^2(\theta)$

$\rightarrow \boldsymbol{cos(2\theta) = 2cos^2(\theta) - 1}$

Also,

$\rightarrow cos(2\theta) = (1 - sin^2(\theta)) - sin^2(\theta) = 1 - sin^2(\theta) - sin^2(\theta)$

$\rightarrow \boldsymbol{cos(2\theta) = 1 - 2sin^2(\theta)}$

DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES

Deriving the **half-angle identities**:

As we know,

$$\boldsymbol{cos(2\theta) = 2cos^2(\theta) - 1}$$

Let $2\theta = \alpha$, so$\theta = \frac{\alpha}{2}$:

$\rightarrow cos(\alpha) = 2cos^2\left(\frac{\alpha}{2}\right) - 1$ or $\frac{1+\cos(\alpha)}{2} = cos^2\left(\frac{\alpha}{2}\right)$

$\rightarrow$ $$\boldsymbol{cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+cos(\alpha)}{2}}}$$ ... (iv)

DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES

Now since,

$$\mathbf{1 - cos(2\theta) = 2sin^2(\theta)}$$

Let $2\theta = \alpha$, so$\theta = \frac{\alpha}{2}$:

$\rightarrow 1 - cos(\alpha) = 2sin^2\left(\frac{\alpha}{2}\right)$ or $\frac{1-\cos(\alpha)}{2} = sin^2\left(\frac{\alpha}{2}\right)$

$\rightarrow$ $$\mathbf{sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-cos(\alpha)}{2}}} \quad \text{... (v)}$$

DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES

Using (iv) and (v),

$$\rightarrow \quad \frac{sin\left(\frac{\alpha}{2}\right)}{cos\left(\frac{\alpha}{2}\right)} = \pm \frac{\sqrt{\frac{1-cos(\alpha)}{2}}}{\sqrt{\frac{1+cos(\alpha)}{2}}}$$

$$\rightarrow \quad tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-cos(\alpha)}{1+cos(\alpha)}} \qquad \text{... (vi)}$$