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Area under a curve

Unit 11 Lesson 6

Students will be able to:

find the areas under graphs of polynomial functions.

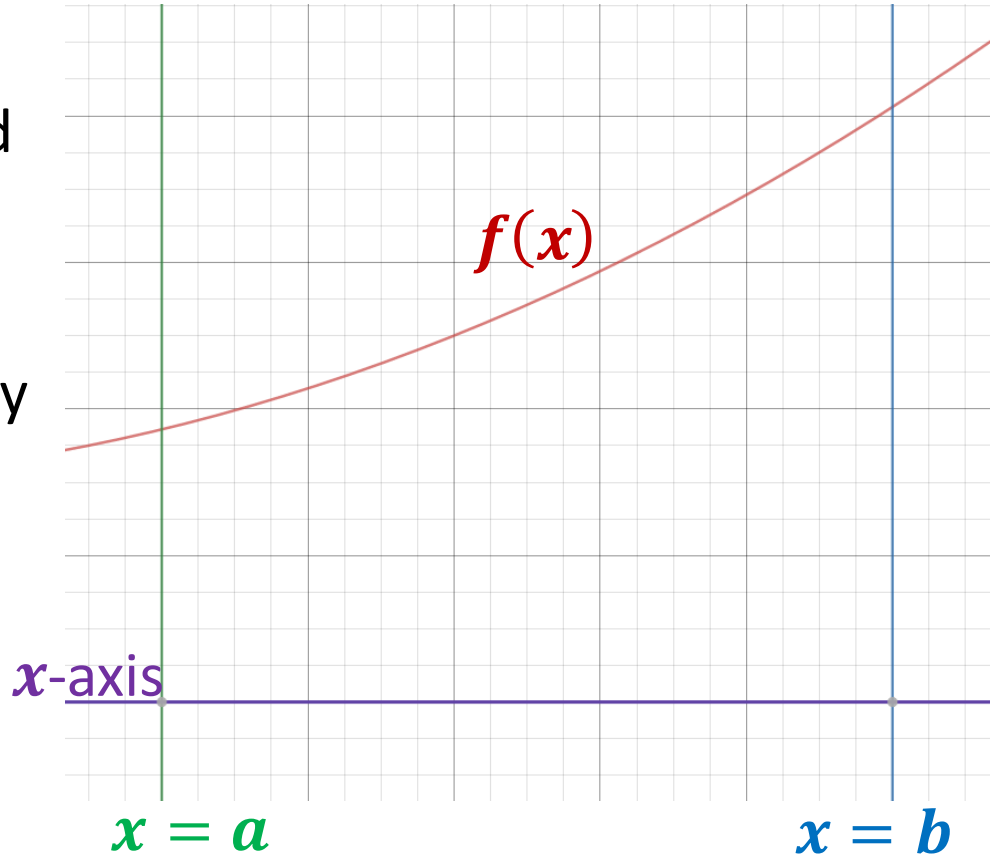
Key Vocabulary:

- Regular Partition
- Left Sum
- Right Sum
- Middle Sum

AREA UNDER A CURVE

Let $f(x)$ be a continuous, non-negative function on the closed interval $[a, b]$.

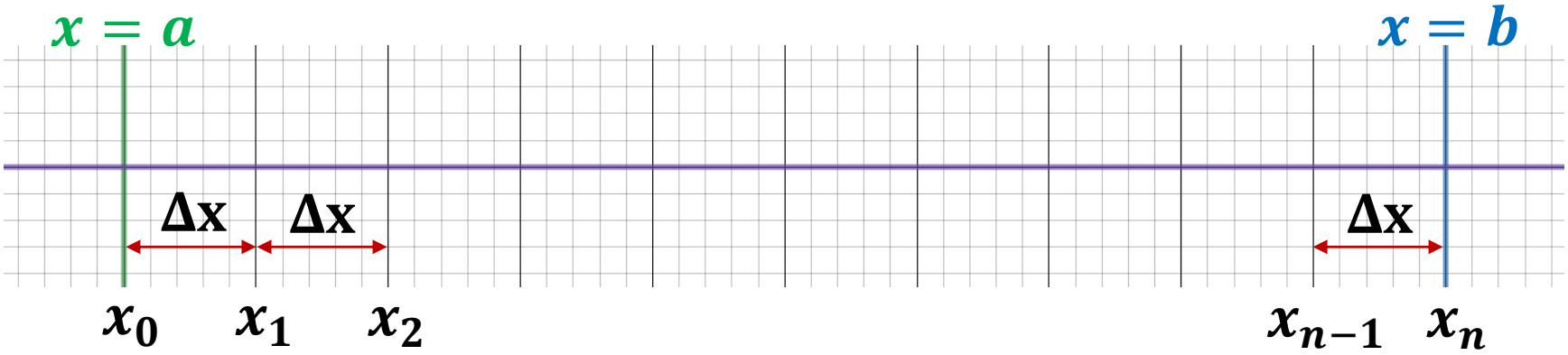
Determine the area bounded by $f(x)$, the vertical lines $x = a$ and $x = b$, and the x-axis.



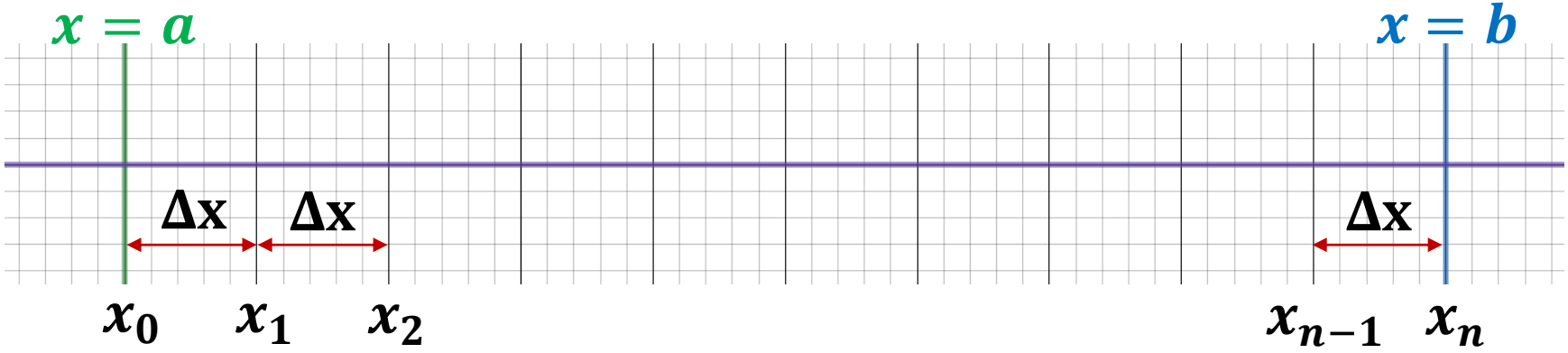
We can use rectangles to approximate the area under the curve by subdividing or partitioning $[a, b]$ into n number of rectangles of same width. Equal width partitions are called **regular partitions**.

$$width = \Delta x = \frac{b - a}{n}$$

where the partition points in terms of $\boldsymbol{a}$ and $\boldsymbol{\Delta x}$:



AREA UNDER A CURVE

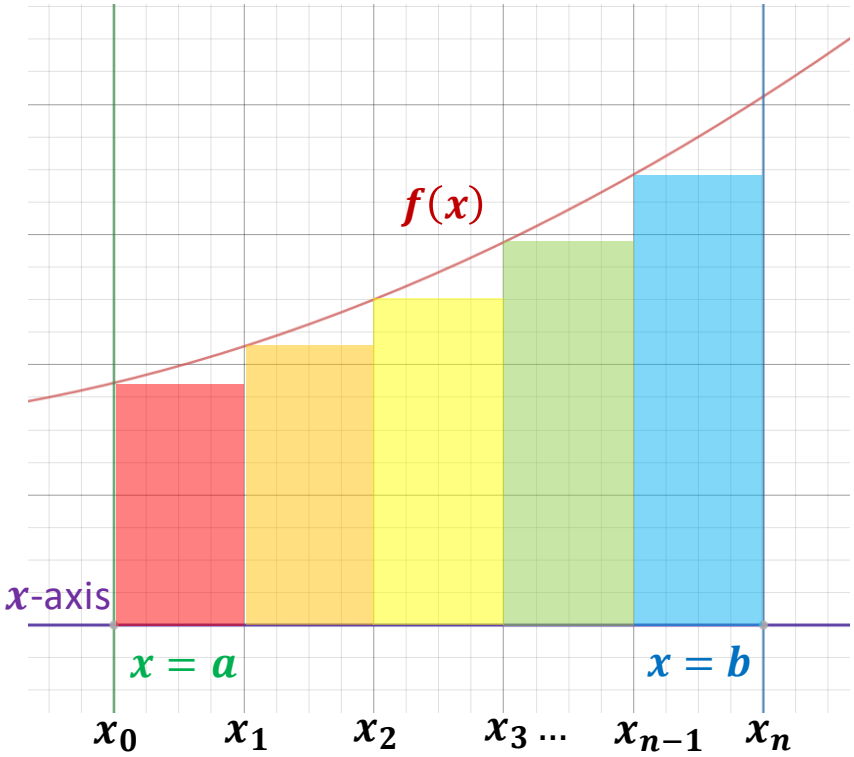


Partition points:

$$x_0 = a + 0 \cdot \Delta x \qquad x_0 = a$$
$$x_1 = a + 1 \cdot \Delta x$$
$$x_2 = a + 2 \cdot \Delta x$$
$$\vdots$$
$$x_{n-1} = a + (n-1) \cdot \Delta x$$
$$x_n = a + n \cdot \Delta x \qquad x_n = b$$

If the rectangles' height are taken from **left side** of the rectangles

Partition	Height	Width (Δx)	Area
$1st$	$f(x_0)$	$x_1 - x_0$	$\Delta x \cdot f(x_0)$
$2nd$	$f(x_1)$	$x_2 - x_1$	$\Delta x \cdot f(x_1)$
$3rd$	$f(x_2)$	$x_3 - x_2$	$\Delta x \cdot f(x_2)$
...	...	...	...
$(n-1)th$	$f(x_{n-2})$	$x_{n-1} - x_{n-2}$	$\Delta x \cdot f(x_{n-2})$
nth	$f(x_{n-1})$	$x_n - x_{n-1}$	$\Delta x \cdot f(x_{n-1})$



The total A_n of the n rectangles is given by the sums of the areas

$$A_n = \left(\frac{b-a}{n}\right) \cdot f(x_0) + \left(\frac{b-a}{n}\right) \cdot f(x_2) + \left(\frac{b-a}{n}\right) \cdot f(x_3) + \cdots + \left(\frac{b-a}{n}\right) \cdot f(x_{n-2}) + \left(\frac{b-a}{n}\right) \cdot f(x_{n-1})$$

$$A_n(left\ sum) = \sum_{n=1}^{n} \left(\frac{b-a}{n}\right) \cdot f(x_{n-1})$$

Sample Problem 1: Let $y = f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. Determine the left sum for regular partition into 4 subintervals.

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$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \Delta x = \frac{1}{2}$$

$$x_{n-1} = a + (n-1) \cdot \Delta x = 0 + (n-1)\left(\frac{1}{2}\right) = x_n = \frac{n-1}{2}$$

Sample Problem 1: Let $y = f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. Determine the left sum for regular partition into 4 subintervals.

$$f(x_{n-1}) = f\left(\frac{n-1}{2}\right) = 1 + \frac{1}{2}\left(\frac{n-1}{2}\right)^2 = 1 + \frac{1}{2}\left(\frac{n^2 - 2n + 1}{4}\right)$$

$$f(x_{n-1}) = \frac{8}{8} + \left(\frac{n^2 - 2n + 1}{8}\right)$$

$$f(x_{n-1}) = \frac{n^2 - 2n + 9}{8}$$

Sample Problem 1: Let $y = f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. Determine the left sum for regular partition into 4 subintervals.

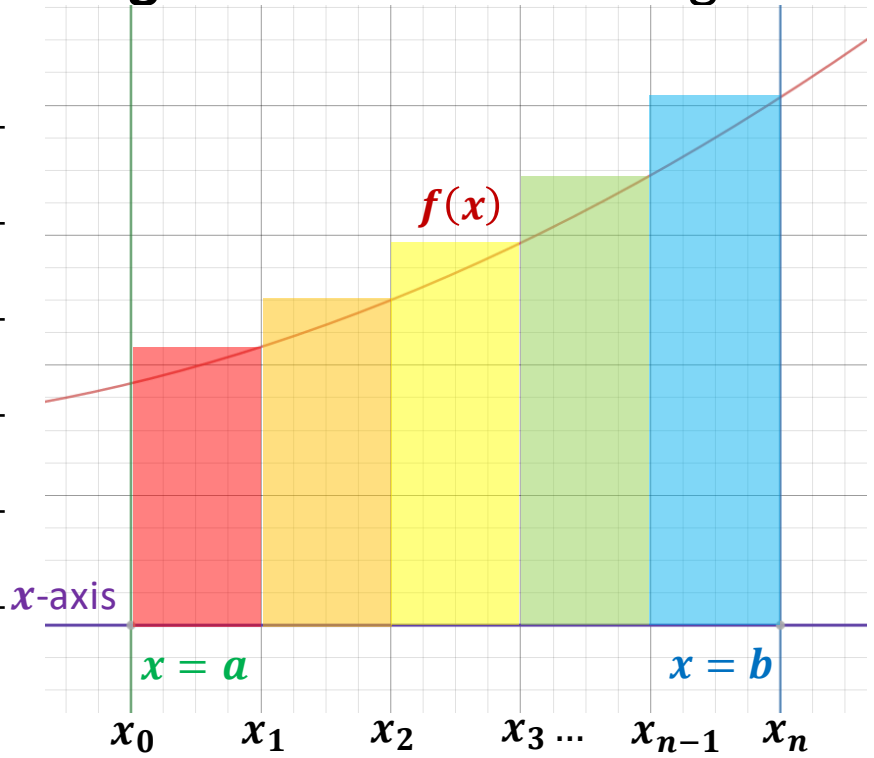
$$A_n(left\ sum) = \sum_{n=1}^{n} \left(\frac{b-a}{n}\right) \cdot f(x_{n-1})$$

$$= \sum_{n=1}^{4} \left(\frac{1}{2}\right) \cdot \left(\frac{n^2 - 2n + 9}{8}\right) = \frac{1}{2} \sum_{n=1}^{4} \left(\frac{n^2 - 2n + 9}{8}\right)$$

$$A_4(left\ sum) = \frac{1}{2}\left(\frac{23}{4}\right) = \frac{23}{8} = 2.875$$

If the rectangles' height are taken from **right side** of the rectangles

Partition	Height	Width (Δx)	Area
$1st$	$f(x_1)$	$x_1 - x_0$	$\Delta x \cdot f(x_0)$
$2nd$	$f(x_2)$	$x_2 - x_1$	$\Delta x \cdot f(x_1)$
$3rd$	$f(x_3)$	$x_3 - x_2$	$\Delta x \cdot f(x_2)$
...	...	...	...
$(n-1)th$	$f(x_{n-1})$	$x_{n-1} - x_{n-2}$	$\Delta x \cdot f(x_{n-1})$
nth	$f(x_n)$	$x_n - x_{n-1}$	$\Delta x \cdot f(x_n)$



The total A_n of the n rectangles is given by the sums of the areas

$$A_n = \left(\frac{b-a}{n}\right) \cdot f(x_0) + \left(\frac{b-a}{n}\right) \cdot f(x_2) + \left(\frac{b-a}{n}\right) \cdot f(x_3) + \cdots + \left(\frac{b-a}{n}\right) \cdot f(x_{n-1}) + \left(\frac{b-a}{n}\right) \cdot f(x_n)$$

$$A_n(right\ sum) = \sum_{n=1}^{n} \left(\frac{b-a}{n}\right) \cdot f(x_n)$$

Sample Problem 1: Let $y = f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. Determine the right sum for regular partition into 4 subintervals.

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$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \Delta x = \frac{1}{2}$$

$$x_n = a + n \cdot \Delta x = 0 + n\left(\frac{1}{2}\right) = x_n = \frac{n}{2}$$

$$f(x_n) = f\left(\frac{n}{2}\right) = 1 + \frac{1}{2}\left(\frac{n}{2}\right)^2 = 1 + \frac{1}{2}\left(\frac{n^2}{4}\right) = f(x_n) = 1 + \frac{n^2}{8}$$

Sample Problem 1: Let $y = f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. Determine the right sum for regular partition into 4 subintervals.

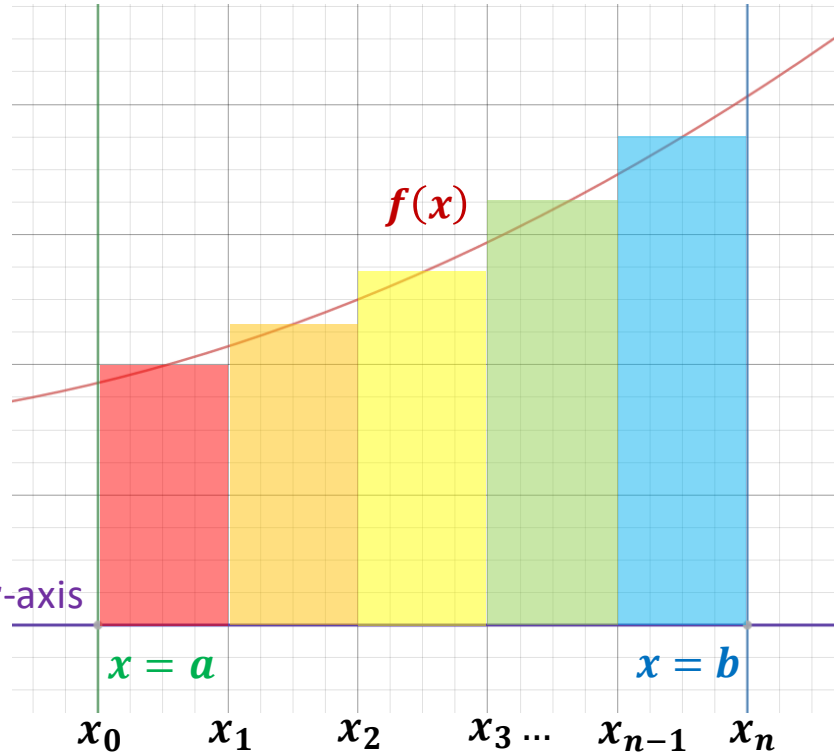
$$A_n(\text{right sum}) = \sum_{n=1}^{n} \left(\frac{b-a}{n}\right) \cdot f(x_n) = \sum_{n=1}^{4} \left(\frac{1}{2}\right) \cdot \left(1 + \frac{n^2}{8}\right)$$

$$= \frac{1}{2} \sum_{n=1}^{4} \left(1 + \frac{n^2}{8}\right) = \frac{1}{2}\left(\frac{31}{4}\right)$$

$$A_4(\text{right sum}) = \frac{31}{8} = 3.875$$

If the rectangles' height are taken from **middle** of the rectangles

Partition	Height	Width (Δx)	Area
$1st$	$f\left(\frac{x_1+x_0}{2}\right)$	x_1-x_0	$\Delta x \cdot f\left(\frac{x_1+x_0}{2}\right)$
$2nd$	$f\left(\frac{x_2+x_1}{2}\right)$	x_2-x_1	$\Delta x \cdot f\left(\frac{x_2+x_1}{2}\right)$
$3rd$	$f\left(\frac{x_3+x_2}{2}\right)$	x_3-x_2	$\Delta x \cdot f\left(\frac{x_3+x_2}{2}\right)$
…	…	…	…
$(n-1)th$	$f\left(\frac{x_{n-2}+x_{n-1}}{2}\right)$	$x_{n-1}-x_{n-2}$	$\Delta x \cdot f\left(\frac{x_{n-2}+x_{n-1}}{2}\right)$
nth	$f\left(\frac{x_n+x_{n-1}}{2}\right)$	x_n-x_{n-1}	$\Delta x \cdot f\left(\frac{x_n+x_{n-1}}{2}\right)$



The total A_n of the n rectangles is given by the sums of the areas

$$A_n = \left(\frac{b-a}{n}\right) \cdot f\left(\frac{x_1 + x_0}{2}\right) + \left(\frac{b-a}{n}\right) \cdot f\left(\frac{x_2 + x_1}{2}\right) + \left(\frac{b-a}{n}\right) \cdot f\left(\frac{x_3 + x_2}{2}\right) + \cdots$$
$$+ \left(\frac{b-a}{n}\right) \cdot f\left(\frac{x_{n-2} + x_{n-1}}{2}\right) + \left(\frac{b-a}{n}\right) \cdot f\left(\frac{x_n + x_{n-1}}{2}\right)$$

$$A_n(middle\ sum) = \sum_{n=1}^{n} \left(\frac{b-a}{n}\right) \cdot f\left(\frac{x_n + x_{n-1}}{2}\right)$$

Sample Problem 3: Let $y = f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. Determine the middle sum for regular partition into 4 subintervals.

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$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \Delta x = \frac{1}{2}$$

$$\frac{x_n + x_{n-1}}{2} = \frac{\frac{n}{2} + \frac{n-1}{2}}{2} = \frac{\frac{n+n-1}{2}}{2} = \frac{\frac{2n-1}{2}}{2} = \frac{2n-1}{2} \cdot \frac{1}{2}$$

$$\frac{x_n + x_{n-1}}{2} = \frac{2n-1}{4}$$

Sample Problem 3: Let $y = f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. Determine the middle sum for regular partition into 4 subintervals.

$$f\left(\frac{x_n + x_{n-1}}{2}\right) = f\left(\frac{2n-1}{4}\right) = 1 + \frac{1}{2}\left(\frac{2n-1}{4}\right)^2$$

$$= 1 + \frac{1}{2}\left(\frac{4n^2 - 4n + 1}{16}\right) = \frac{32}{32} + \left(\frac{4n^2 - 4n + 1}{32}\right)$$

$$f\left(\frac{x_n + x_{n-1}}{2}\right) = \frac{4n^2 - 4n + 33}{32}$$

Sample Problem 3: Let $y = f(x) = 1 + \frac{1}{2}x^2$ on $[0, 2]$. Determine the middle sum for regular partition into 4 subintervals.

$$A_n(middle\ sum) = \sum_{n=1}^{n} \left(\frac{b-a}{n}\right) \cdot f\left(\frac{x_n + x_{n-1}}{2}\right)$$

$$= \sum_{n=1}^{4} \left(\frac{1}{2}\right) \cdot \left(\frac{4n^2 - 4n + 33}{32}\right) = \frac{1}{2} \sum_{n=1}^{4} \left(\frac{4n^2 - 4n + 33}{32}\right)$$

$$A_4(middle\ sum) = \frac{1}{2}\left(\frac{53}{8}\right) = \frac{53}{16} = 3.3125$$